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THERMAL INTERACTION OF TWO STREAMS IN BOUNDARY-LAYER FLOW SEPARATED BY A PLATE

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Abstract—The problem of heat exchange between two fluid streams in boundary layer flow separated by a flat plate is considered. A general analysis applicable to cocurrent or countercurrent, laminar or turbulent flow is presented. An exact solution for the temperature distribution and the heat transfer along the plate is obtained for the special case of constant property, cocurrent, inviscid flow. In the less-restrictive case of constant property laminar or turbulent flow, the wall temperature and heat flux are predicted using the method of superposition for which results of a desired degree of accuracy are possible. For the most general case of variable physical properties the finite difference solution of the momentum and energy equations in von Mises form is indicated. Finally, some illustrative results for cocurrent, constant property, laminar flow in the streams are reported. It is shown that heat exchange analyses which neglect the thermal interaction between the fluid streams could be in serious error.

NOMENCLATURE

a, parameter defined as α/u_∞;
b, thickness of the plate;
C, constant in equation (18);
c_p, specific heat at constant pressure;
d, parameter defined as $(k_1 a_1^{-\frac{1}{2}})/[k_1 a_1^{-\frac{1}{2}} + k_2 a_2^{-\frac{1}{2}}] = (k_w/b)(1/g_2 k_2)$;
g_1, parameter defined as $(k_w/b)(k_1 a_1^{-\frac{1}{2}} + k_2 a_2^{-\frac{1}{2}})/(k_1 k_2 a_2^{-\frac{1}{2}})$;
g_2, parameter defined as $g_1(a_1/a_2)^{\frac{1}{2}}$;
h, local heat-transfer coefficient;
k, thermal conductivity;
L, plate length;
M, parameter defined as $[C_2 k_2 Pr_2^{n_2}(u_{2\infty}v_1)^{m_2}/C_1 k_1 Pr_1^{n_1}(u_{1\infty}v_2)^{m_2}]$;
m, exponent in equation (18);
n, exponent in equation (18);
P, parameter defined as $C_1(k_1/k_w)(u_{1\infty}/v_1)Pr_1^{n_1}$;
Pr, Prandtl number, $\mu c_p/k$;
q, heat flux;
Re_x, local Reynolds number, $u_\infty x/v$;
S_{ik}, temperature slope, $d\theta_i/d\xi$, on kth subinterval of plate;
T, temperature;
u, velocity in the x-direction;
v, velocity in the y-direction;
x, coordinate in the direction of flow;
x_i^*, dimensionless coordinate defined as $u_{1\infty}x_i/v_1$, for cocurrent flow $x_2^* = x_1^*$ and for counterflow $x_2^* = Re_{1,L} - x_1^*$;
$x_{i,j}^*$, location of the far side of the jth subinterval as measured in the i coordinate system;
y, coordinate transverse to the direction of flow;
α, thermal diffusivity, $k/\rho c_p$;
β, exponent in equation (18);
γ, exponent in equation (18);
δ, subinterval length;
ε_h, eddy diffusivity of heat;
ε_m, eddy diffusivity of momentum;
ζ, parameter defined as $(\alpha x/u_\infty)^{\frac{1}{2}}$;
θ_i, dimensionless temperature defined as $(T_i - T_{2\infty})/(T_{1\infty} - T_{2\infty})$;
μ, dynamic viscosity;
v, kinematic viscosity;
ρ, density;
ξ, position along plate surface;
ψ, stream function;

τ, shear stress.

Subscripts
- i, refers to either stream 1 or 2;
- L, based upon plate length;
- Q, based upon uniform heat flux boundary condition;
- T, based upon uniform temperature boundary condition;
- x, based upon distance from leading edge;
- w, refers to the wall;
- 1, refers to stream 1;
- 2, refers to stream 2;
- ∞, refers to the free stream.

INTRODUCTION

APPLICATIONS of heat exchange between two fluid streams separated by a plane wall in boundary layer flow are extremely common in practice, yet fundamental analysis of the transport process occurring in such situations has received little attention. Basic models of heat transfer across a solid wall consider only one stream at a time, usually under the assumption of a constant wall temperature or heat flux and sometimes for a prescribed wall temperature or heat flux [1, 2]. The primary purpose of this paper is to examine the validity and accuracy of such heat transfer predictions, i.e. those which neglect the interaction between the streams.

Of particular concern for the analysis and design of heat transfer equipment is the overall heat-transfer coefficient which is the addition in series of individual thermal resistances, each measured or predicted in the absence of the resistance in the other stream. The overall heat-transfer coefficient is given by [3]

$$U = 1/[1/h_1 + b/k_w + 1/h_2]. \qquad (1)$$

In essentially all practical heat transfer processes the local heat-transfer coefficients can logically be expected to vary with surface position. The temperature at all points along the solid–fluid interface must therefore also vary even though the free stream temperatures remain constant. Traditionally, the equipment designer would select independent film coefficient correlations $h_1(x)$ and $h_2(x)$, and then use equation (1) to calculate the local heat flux

$$q(x) = U(x)(T_{1\infty} - T_{2\infty}). \qquad (2)$$

As will be shown later on, this procedure leads to error since the correlations $h_1(x)$ and $h_2(x)$ seldom (if ever) are applicable to the surface temperature variations actually encountered in the given piece of equipment. The main theme running through the remaining sections of this paper is that the traditional practice of neglecting the interaction between the fluid streams leads to error in engineering analysis and design, and the work presented in this report is concerned with ascertaining this error. (In mass-transfer operations there is the analogous practice of adding interphase resistances in defining an overall mass-transfer coefficient. Here too, as pointed out in [4],* such practice is not always valid.)

The problem is formulated in general terms and several fluid models are considered: an inviscid fluid, a constant property viscous fluid, and a variable property viscous fluid. For a range of parameters of interest, heat-transfer rates for the constant fluid property model are compared to the corresponding rates obtained by neglecting the interaction between the streams.

A review of literature has failed to reveal any studies dealing with the problem. The similarity solution for laminar flow reported by Kuznetsov [5] is a limiting case of the more general problem investigated in this paper. The case studied by Kuznetsov, where one surface of the plate is maintained at a constant temperature, corresponds physically to the case of infinitely large heat-transfer coefficient on that surface.

PHYSICAL MODEL AND FORMULATION OF PROBLEM

A schematic description of the system con-

* This reference was brought to the authors' attention by one of the reviewers.

sisting of two streams, either of which is in laminar or turbulent flow, is given in Fig. 1. The streams designated by 1 and 2 are shown in cocurrent flow in Fig. 1a and in counterflow in Fig. 1b. The flow and heat transfer are assumed to be steady, two dimensional, with negligible viscous heat dissipation and zero pressure gradients. The initial velocity and temperature for either stream are taken to be uniform, and conduction in the plate is considered to be one-dimensional, in the transverse direction only. Further, assuming that the conventional boundary layer approximations are applicable to the present problem, the conservation equations of mass momentum and energy for either stream are written respectively as

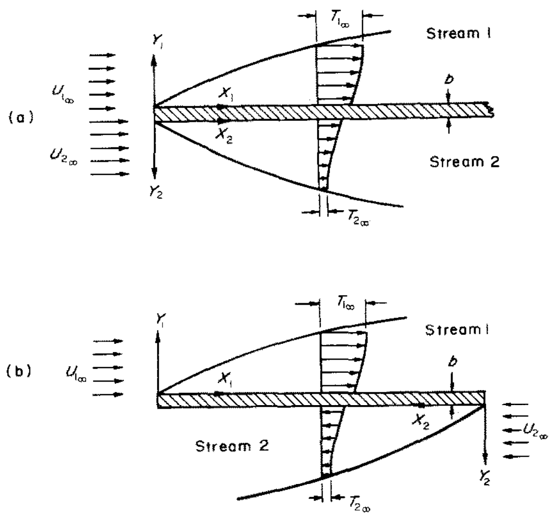

FIG. 1. Schematic diagram of physical models considered: (a) Cocurrent flow; (b) Counterflow.

$$\frac{\partial(\rho_i u_i)}{\partial x_i} + \frac{\partial(\rho_i v_i)}{\partial y_i} = 0, \tag{3}$$

$$\rho_i\left(u_i\frac{\partial u_i}{\partial x_i} + v_i\frac{\partial u_i}{\partial y_i}\right) = \frac{\partial}{\partial y_i}\left[(\mu_i + \rho_i\varepsilon_{m,i})\frac{\partial u_i}{\partial y_i}\right], \tag{4}$$

and

$$\rho_i c_{pi}\left(u_i\frac{\partial T_i}{\partial x_i} + v_i\frac{\partial T_i}{\partial y_i}\right) = \frac{\partial}{\partial y_i}\left[(k_i + \rho_i c_{pi}\varepsilon_{h,i})\frac{\partial T_i}{\partial y_i}\right] \tag{5}$$

with $i = 1, 2$. The initial, boundary, and interface conditions are

$$u_i = u_{i\infty} \text{ at } x_i = 0, \tag{6a}$$

$$v_i = u_i = 0 \text{ at } y_i = 0, \tag{6b}$$

$$u_i \to u_{i\infty} \text{ as } y_i \to \infty, \tag{6c}$$

$$T_i = T_{i\infty} \text{ at } x_i = 0, \tag{7a}$$

$$T_i = T_{iw} \text{ at } y_i = 0, \tag{7b}$$

$$T_i \to T_{i\infty} \text{ as } y_i \to \infty, \tag{7c}$$

and

$$k_2(\partial T_2/\partial y_2)|_{y_2=0} = -k_1(\partial T_1/\partial y_1)|_{y_1=0} = (k_w/b)\left[T_{2w}(x_2) - T_{1w}(x_1)\right]. \tag{7d}$$

In the last expression, $x_2 = x_1$ for cocurrent flow, and $x_2 = (L - x_1)$ for counterflow.

There is no possibility of obtaining a similarity solution of the conservation equation (3)–(5) with the boundary conditions (6, 7) because of the interaction between the two streams. In principle, a finite-difference numerical solution can be obtained as indicated later in connection with the variable property fluid model.

INVISCID CONSTANT PROPERTY FLOW MODEL

Considerable insight into the heat transfer process can be obtained by assuming inviscid, constant property flow in both streams. Physically, this means that the velocity boundary layer is much thinner than the temperature boundary layer. The temperature distribution to a good approximation can then be predicted on the assumption that in each fluid region, the velocity is everywhere equal to the respective free stream velocity. The boundary-layer equations (3)–(5) then simplify to the following form:

$$u_{i\infty}\frac{\partial T_i}{\partial x_i} = \alpha_i\frac{\partial^2 T_i}{\partial y_i^2}, \; i = 1, 2. \tag{8}$$

This model would apply to cases of heat transfer where the Prandtl number is much less than 1·0 [1, 6], however, the utility of the model in the past has been that it yields a closed form solution and predicts correct trends. It has been demonstrated that in similar mass-transfer problems the inviscid flow theory is equivalent to the penetration film theory [7].

A mathematical problem analogous to the one defined by equation (8) with the boundary conditions (7) has already been solved. Omitting all of the details, one can show that for the case of cocurrent flow, the temperature distributions in the two streams are given by [8]

$$\theta_1(x_1, y_1) = d\left\{1 + \left(\frac{k_2 g_2}{k_1 g_1}\right) \times \left[\operatorname{erf}(y_1/2\zeta_1) + \exp(g_1 y_1 + g_1^2\zeta_1^2) \times \operatorname{erfc}(y_1/2\zeta_1) + g_1\zeta_1\right]\right\} \tag{9}$$

and

$$\theta_2(x_2, y_2) = g\{\operatorname{erfc}(y_2/2\zeta_2) - \exp(g_2 y_2 + g_2^2\zeta_2^2) \times \operatorname{erfc}[(y_2/2\zeta_2) + g_2\zeta_2]\}. \tag{10}$$

By setting $y_1 = y_2 = 0$ in equations (9) and (10), and making use of the identities $g_1\zeta_1 = g_2\zeta_2 \equiv g\zeta$ and $q_1 = -q_2 \equiv q$, one obtains the local wall heat flux

$$q/[k_w/b(T_{2\infty} - T_{1\infty})] = \exp(g^2\zeta^2)\operatorname{erfc}(g\zeta) \tag{11}$$

and the local wall temperature difference

$$\theta_1(x_1, 0) - \theta_2(x_2, 0) = \exp(g^2\zeta^2)\operatorname{erfc}(g\zeta). \tag{12}$$

As expected, the dimensionless heat flux [the left-hand side of (11)] is just equal to the dimensionless temperature difference.

If the convective heat-transfer coefficient $h_i(x_i)$ is defined in the conventional way, i.e.

$$h_i(x_i) = \frac{q_i(x_i)}{[T_i(x_i, 0) - T_{i\infty}]} = \frac{-(k_i\,\partial T_i/\partial y_i)|_{y_i=0}}{[T_i(x_i, 0) - T_{i\infty}]}, \tag{13}$$

it follows from equations (9) and (10) that the heat-transfer coefficients are given by

$$h_1(x_1) = \frac{(k_w/b)\exp(g_1^2\zeta_1^2)\operatorname{erfc}(g_1\zeta_1)}{1 - d\left[1 + \left(\frac{k_2 g_2}{k_1 g_1}\right)\exp(g_1^2\zeta_1^2)\operatorname{erfc}(g_1\zeta_1)\right]} \tag{14}$$

and

$$h_2(x_2) = \frac{(k_w/b)\exp(g_2^2\zeta_2^2)\operatorname{erfc}(g_2\zeta_2)}{d[1 - \exp(g_2^2\zeta_2^2)\operatorname{erfc}(g_2\zeta_2)]}. \tag{15}$$

Examination of equations (14) and (15) shows that the heat-transfer coefficients at the two interfaces are affected by the thermal interaction between the two streams as well as the wall through the parameters d, g_1, g_2 and k_w/b.

VISCOUS CONSTANT PROPERTY FLOW MODEL

Assuming constant fluid properties and temperature independent eddy diffusivity, the energy equation (5), becomes linear, and consequently the method of superposition can be used to express the local wall heat flux. Derivation of the superposition method is not presented here (see [1] and [2]), but is should be noted that the method is mathematically rigorous. Thus, following the method of superposition, wall heat fluxes in regions 1 and 2 are written respectively as

$$q_1(x_1) = \int_0^{x_1} h_1(\xi, x_1)\left(\frac{dT_{1w}}{d\xi}\right)d\xi + \sum_{j=1}^{n} h_1(\xi_j, x_1)\,\Delta T_{1w,j} \tag{16}$$

and

$$q_2(x_2) = \int_0^{x_2} h_2(\xi, x_2)\left(\frac{dT_{2w}}{d\xi}\right)d\xi + \sum_{j=1}^{n} h_2(\xi_j, x_2)\,\Delta T_{2w,j} \tag{17}$$

where $h_i(\xi, x_i)$ denotes the heat-transfer coefficient at x_i caused by a discontinuity in wall temperature (finite-sized or infinitesimal) occurring at $0 < \xi < x_i$; and $\Delta T_{iw,j}$ denotes the magnitude the jth finite-sized wall temperature discontinuity. If the quantity $[T_{i\infty} - T_{iw}(x)]$ is a continuous function of position for $x \geqslant 0$, which is the situation in the present problem, the summations in equations (16) and (17) are identically zero. The following analysis employs equations (16) and (17) to determine the temperature and heat flux distributions along the wall.

In many physically important problems $h(\xi, x)$ can be expressed as [9]

$$h(x, \xi) = C(k/x)\, Re_x^m Pr^n \left[1 - (\xi/x)^\gamma\right]^\beta \qquad (18)$$

where the constants C, m, n, γ and β are given in Table 1 for several flow situations.

Table 1. Constants used in equation (18) *for various flow situations* [9]

	C	γ	β	m	n
(1) Laminar flow, $Pr > 1$	0·332	$\frac{3}{4}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
(2) Turbulent flow, $Pr > 1$	0·029	$\frac{9}{10}$	$-\frac{1}{9}$	$\frac{4}{5}$	$\frac{1}{2}$
(3) Laminar flow of liquid metals, $u = u_\infty$	0·564	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Substituting equation (18) into equations (16) and (17), and realizing that wall temperature is continuous, we have that

$$q_1(x_1) = C_1(k_1/x_1)\, Re_{x_1}^{m_1} Pr_1^{n_1} \int_0^{x_1} \times \left[1 - (\xi/x_1)^{\gamma_1}\right]^{\beta_1} (\mathrm{d}T_{1w}/\mathrm{d}\xi)\,\mathrm{d}\xi \qquad (19)$$

and

$$q_2(x_2) = C_2(k_2/x_2)\, Re_{x_2}^{m_2} Pr_2^{n_2} \int_0^{x_2} \times \left[1 - (\xi/x_2)^{\gamma_2}\right]^{\beta_2} (\mathrm{d}T_{2w}/\mathrm{d}\xi)\,\mathrm{d}\xi. \qquad (20)$$

Substituting equations (19) and (20) into the interface conditions (7d) and introducing the dimensionless variables

$$x_1^* = (u_{1\infty} x_1/\nu_1); \qquad x_2^* = (u_{1\infty} x_2/\nu_1);$$
$$\theta_i = (T_{iw} - T_{2\infty})/(T_{1\infty} - T_{2\infty}) \qquad (21)$$

yields

$$-\left[M(x_1^*)^{1-m_1}/(x_2^*)^{1-m_2}\right] \int_0^{x_2^*} \times \left[1 - (\xi/x_2^*)^{\gamma_2}\right]^{\beta_2} (\mathrm{d}\theta_2/\mathrm{d}\xi)\,\mathrm{d}\xi = \int_0^{x_1^*} \left[1 - (\xi/x_1^*)^{\gamma_1}\right]^{\beta_1} (\mathrm{d}\theta_1/\mathrm{d}\xi)\,\mathrm{d}\xi \qquad (22)$$

and

$$\left[P/(x_1^*)^{1-m_1}\right] \int_0^{x_1^*} \left[1 - (\xi/x_1^*)^{\gamma_1}\right]^{\beta_1} \times (\mathrm{d}\theta_1/\mathrm{d}\xi)\,\mathrm{d}\xi = \theta_2(x_2^*) - \theta_1(x_1^*). \qquad (23)$$

It is noted that for cocurrent flow $x_2^* = x_1^*$ and for counterflow $x_2^* = Re_{1,L} - x_1^*$.

Solution of equations (22) and (23) is accomplished in the following manner. The interval $0 < x_1^* < Re_{1,L}$ is broken up into N subintervals of length δ_k, which are sufficiently small, so that on every subinterval the derivatives $\mathrm{d}\theta_i/\mathrm{d}\xi$ may be taken as constants ([1], p. 182). It is clear that such an approximation becomes increasingly accurate as subinterval size is reduced. Letting $s_{i,k}$ denote $\mathrm{d}\theta_i/\mathrm{d}\xi$ on the kth subinterval,

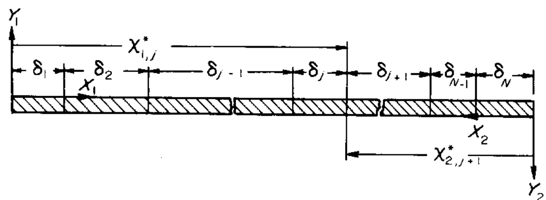

FIG. 2. Illustration of the notation used in the constant fluid property analysis.

($k = 1, \ldots, N$), equations (22) and (23) can then be rewritten as $2N$ linear algebraic equation whose solution yields the $2N$ quantities $s_{i,k}$. Referring to Fig. 2 for the indexing system, the two algebraic equations for the position $x_{1,j}^*$ are thus

$$-\left[M(x_{1,j}^*)^{1-m_1}/(x_{2,t}^*)^{1-m_2}\right] \sum_{k=q}^{t} s_{2,k} \int_{x_{2,k}^*-\delta_k}^{x_{2,k}^*} \times \left[1 - (\xi/x_{2,t}^*)^{\gamma_2}\right]^{\beta_2} \mathrm{d}\xi$$
$$= \sum_{k=1}^{j} s_{1,k} \int_{x_{1,k}^*-\delta_k}^{x_{1,k}^*} \left[1 - (\xi/x_{1,t}^*)^{\gamma_1}\right]^{\beta_1} \mathrm{d}\xi, \qquad (24)$$

and

$$[P/(x_1^*)^{1-m_1}] \sum_{k=1}^{j} s_{1,k} \int_{x_{1,k}^*-\delta_k}^{x_{1,k}^*} [1-(\xi/x_{1,j}^*)^{\gamma_1}]^{\beta_1} \mathrm{d}\xi = \sum_{k=q}^{t} s_{2,k}\delta_k - \sum_{k=1}^{j} s_{1,k}\delta_k - 1. \quad (25)$$

With $s_{i,k}$ known, the wall temperature at the general position $x_{i,j}^*$ is given by

$$\theta_1(x_{1,j}^*) = 1 + \sum_{k=1}^{j} s_{1,k}\delta_k, \quad (26a)$$

and

$$\theta_2(x_{2,t}^*) = \sum_{k=q}^{t} s_{2,k}\delta_k. \quad (26b)$$

In equations (24)–(26) $t = j$, $q = 1$, and $x_{2,j}^* = x_{1,j}^*$ for cocurrent flow, while for counterflow $t = j+1$, $q = N$ and $x_{2,j+1}^* = Re_{1,L} - x_{1,j}^*$. Heat flux is obtained from the dimensionless form of equation (7d).

By assuming that the flow is cocurrent and that the same type of flow exists on each side of the plate, it can be shown from equations (24) and (25) that

$$s_{1,j} = \frac{M(P/x_j^{*\,1-m}) \sum_{k=1}^{j-1} s_{1,k} \int_{x_{k-1}^*}^{x_k^*} [1-(\xi/x_j^*)^{\gamma}]^{\beta} \mathrm{d}\xi + \theta_1(x_{j-1}^*) - \theta_2(x_{j-1}^*)}{-\delta_j - M(\delta_j + (P/x_j^{*\,1-m}) \int_{x_{j-1}^*}^{x_j^*} [1-(\xi/x_j^*)^{\gamma}]^{\beta} \mathrm{d}\xi)} \quad (27)$$

and

$$s_{2,j} = -s_{1,j}/M. \quad (28)$$

From a computational standpoint, the solution is simpler for the cocurrent than for the counterflow flow configuration. The simplification is due to the fact that the slopes $s_{i,k}$ can be computed recursively, i.e. starting at $j = 1$ with the conditions $\theta_1(x_0^*) = 1$ and $\theta_2(x_0^*) = 0$, the $s_{i,j}$ can be calculated for all successive j's, while a matrix inversion is required when equations (24) and (25) are used for counterflow.

VARIABLE PROPERTY VISCOUS FLOW MODEL

When thermophysical properties are state dependent, solution to the problem can only be obtained by solving simultaneously equations (3)–(5). As a first step in such a solution, equations (4) and (5) are simplified by employing the von Mises transformation [10] obtain respectively

$$\left(\frac{\partial u_i}{\partial x_i}\right)_{\psi_i} = \frac{1}{\rho_{i\infty}^2}\frac{\partial}{\partial \psi_i}\left[\rho_i(\mu_i + \rho_i\varepsilon_{m,i})\,u_i\left(\frac{\partial u_i}{\partial \psi_i}\right)_{x_i}\right]_{x_i} \quad (29)$$

and

$$\left(\frac{\partial T_i}{\partial x_i}\right)_{\psi_i} = \frac{1}{c_{pi}\rho_{i\infty}^2}\frac{\partial}{\partial \psi_i}\left[\rho_i(k_i + \rho_i c_{p_i}\varepsilon_{n,i})u_i\left(\frac{\partial T_i}{\partial \psi_i}\right)_{x_i}\right]_{x_i} \quad (30)$$

where ψ_i is the stream function, taken to be zero at the plate surfaces, and defined so that

$$u_i = \frac{\rho_{i\infty}}{\rho_i}\frac{\partial \psi_i}{\partial y_i} \quad \text{and} \quad v_i = \frac{-\rho_{i\infty}}{\rho_i}\frac{\partial \psi_i}{\partial x_i}. \quad (31)$$

It is noted that the continuity equation (3) is now automatically satisfied. Equations (29) and (30) are subject to the following boundary and interface conditions:

$$\text{as } \psi_i \to \infty: \quad T_i(x_i, \psi_i) \to T_{i\infty}, \quad \text{and} \quad u_i(x_i, \psi_i) \to u_{i\infty}, \quad (32a)$$

$$\text{at } x_i = 0: \quad T_i(0, \psi_i) = T_{i\infty}, \quad \text{and} \quad u_i(0, \psi_i) = u_{i\infty}, \quad (32b)$$

$$\text{at } \psi_i = 0: \quad T_i(x_i, 0) = T_{iw}(x_i). \quad (32c)$$

The interface conditions corresponding to equation (7d) have to be treated carefully, since, under the von Mises transformation, the continuity of the conductive fluxes, embodied in the first part of equation (7d) and expressed as

$$-\frac{\rho_{2\infty}}{\rho_{1\infty}}\frac{(k_1\rho_1)|_{\psi_1=0}}{(k_2\rho_2)|_{\psi_2=0}}\left(u_1\frac{\partial T_1}{\partial\psi_1}\right)\Bigg|_{\psi_1=0} = \left(u_2\frac{\partial T_2}{\partial\psi_2}\right)\Bigg|_{\psi_2=0} \tag{33}$$

is seen to be trivial because

$$u_1|_{\psi_1=0} = u_2|_{\psi_2=0} = 0.$$

This difficulty is removed however, by applying L'Hôpital's rule with the result

$$-\frac{\rho_{2\infty}}{\rho_{1\infty}}\frac{(k_1\rho_1)|_{\psi_1=0}}{(k_2\rho_2)|_{\psi_2=0}}\times\frac{\tau_{w1}(x_1)\mu_{w2}}{\tau_{w2}(x_2)\mu_{w1}}\frac{\partial T_1}{\partial\psi_1}\Bigg|_{\psi_2=0} = \frac{\partial T_2}{\partial\psi_2}\Bigg|_{\psi_2=0}. \tag{34}$$

The additional requirement, corresponding to the latter part of equation (7d), is transformed into

$$(k_w/b)\left[T_{2w}(x_2) - T_{1w}(x_1)\right] = -\frac{1}{\rho_{1\infty}}\,k_1\rho_1u_1\frac{\partial T_1}{\partial\psi_1}\Bigg|_{\psi_1=0} = -\frac{1}{\rho_{1\infty}}(k_1\rho_1)|_{\psi_1=0}\times\lim_{\psi_1\to 0}\left[\frac{u_1(x_1,\psi_1)\left[T_1(x_1,\psi_1) - T_1(x_1,0)\right]}{\psi_1}\right]. \tag{35}$$

On physical grounds, a limit of the bracketed quantity must exist if the rate of heat transfer at the wall is finite.

Equations (29)–(35) completely define the flow and heat transfer in regions 1 and 2. A numerical finite difference solution of an analogous one-region problem has been obtained in [11].

RESULTS AND DISCUSSION

As an illustration, the wall temperature and heat flux variation along the plate are predicted for constant property, cocurrent, laminar flow of streams 1 and 2 using equations (27) and (28). The appropriate constants C, m, n, β and γ for fluids with $1 < Pr < 15$ are given in Table 1. Inspection of equations (22) and (23) reveals that the wall temperature distribution depends on the dimensionless parameters M, P and x_1^*. Physically, the parameter M is a measure of the heat-transfer conductance of stream 2 compared to stream 1, and the parameter P is a measure of the heat-transfer conductance of stream 1 compared with that of the plate. The dimensionless distance x_1^* is just the local Reynolds number in stream 1 based on x_1 as the characteristic dimension.

Typical variation of the local wall temperature is shown in Fig. 3. It is noted that for any value of the parameter M, a decrease in parameter P and/or an increase in Re_{x_1} results in a definite limit of θ_{iw}. It can be proven that this limit is given by $1/(1 + M)$.

The variation of the dimensionless local heat flux is shown in Fig. 4. For fixed parameters P and M heat-transfer rate decreases as Re_{x_1} is increased, an expected result since the laminar boundary layer becomes thicker and increases the resistance to heat transfer. All curves in Fig. 4 approach unity as $Re_{x_1} \to 0$. This behavior is a consequence of neglecting axial heat conduction in the energy equation (5).

For a given P and Re_{x_1}, an increase in M results in an increased heat flux. Such behavior is expected considering the physical significance of that parameter. However, when M is already large, most of the resistance to heat transfer is in stream 1 and the plate so that further increase in M has little effect upon heat transfer. This is demonstrated in Fig. 4 by the closeness of the results for $M = 10$ and 100. Furthermore, from the physical interpretation of P, it is also evident that large P implies large heat transfer. The two dashed curves of Fig. 4, representing heat transfer predicted by the inviscid flow

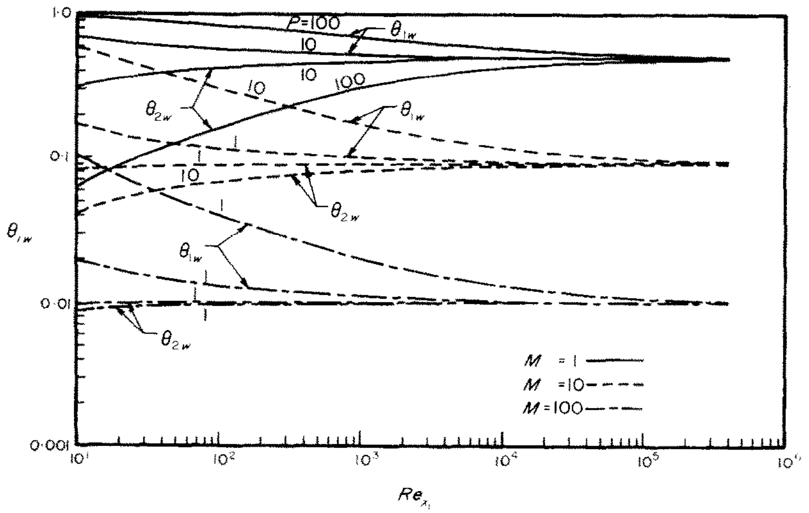

FIG. 3. Local wall temperatures for constant property fluids in laminar, cocurrent flow.

model, are obtained from equation (11) after expressing $g\zeta$ in terms of the parameters P, M and Re_{x_1}. The required expression for $g\zeta$ is readily shown to be

$$g\zeta = 0{\cdot}564/P(1/M + 1)Re_{x_1}^{\frac{1}{2}}. \tag{36}$$

It is seen that the curves based upon the inviscid model are in best agreement with the corresponding viscous results when P is large and in the vicinity of the leading edge, i.e. where the velocity boundary layer is relatively thin.

The seriousness of neglecting the actual wall temperature variation in the heat exchange analysis is illustrated in Fig. 5 which is a comparison of the heat-transfer coefficients predicted by the analysis with those based on

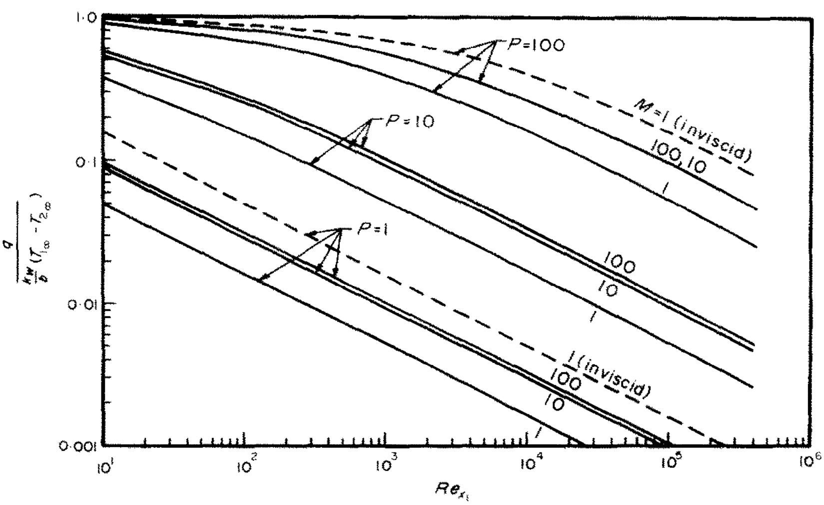

FIG. 4. Effect of parameters M and P upon local heat exchange in laminar, cocurrent flow of constant property fluids.

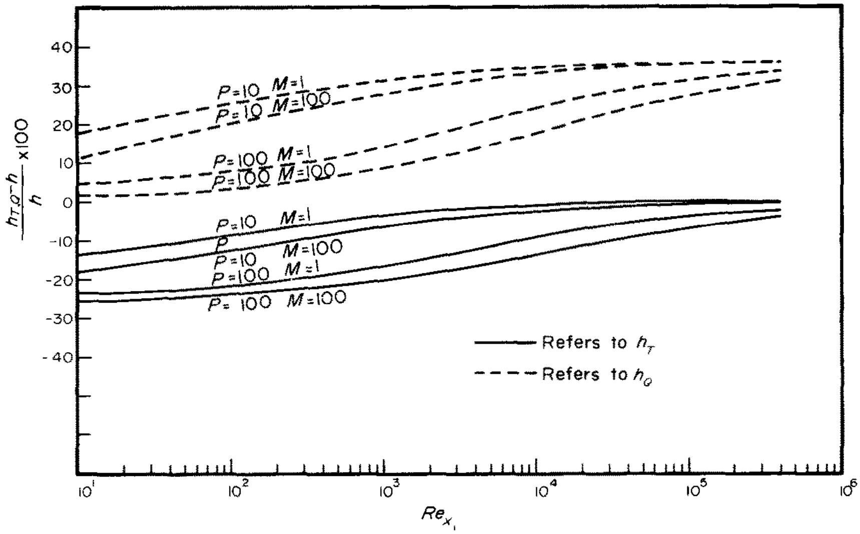

FIG. 5. Comparison of local heat transfer coefficients for laminar, cocurrent flow of constant property fluids.

uniform wall temperature and heat flux boundary conditions. The local heat-transfer coefficients for the uniform wall temperature and uniform heat flux boundary conditions in laminar flow are given respectively by [2]

$$h_{Tx} = 0{\cdot}332\,(k/x)\,Pr^{\frac{1}{3}}\,Re_x^{\frac{1}{2}} \tag{37}$$

and

$$h_{Qx} = 0{\cdot}453\,(k/x)\,Pr^{\frac{1}{3}}\,Re_x^{\frac{1}{2}}. \tag{38}$$

The actual heat-transfer coefficient, h, is calculated from (13), and the quantity $(h_{T,Q} - h)/h$ is found to be independent of the stream. This is always the case in the cocurrent configuration if the same type of flow exists on both sides of the plate. Referring to the physical interpretations of P and M, it is seen that a uniform heat flux boundary condition is approached as M and P both become large and Re_{x_1} becomes small, while a uniform wall temperature condition is approached as M and P both become small and Re_{x_1} becomes large. This explains why in Fig. 5 the results based upon h_{Qx} are in better agreement with the present predictions than those based on h_{Tx} for the parameters $M = P = 100$ and small Re_{x_1}.

A comparison of heat fluxes are shown in Fig. 6. It is seen that as P becomes very large, implying that the plate is nearly adiabatic, the results based on equations (37) and (38)

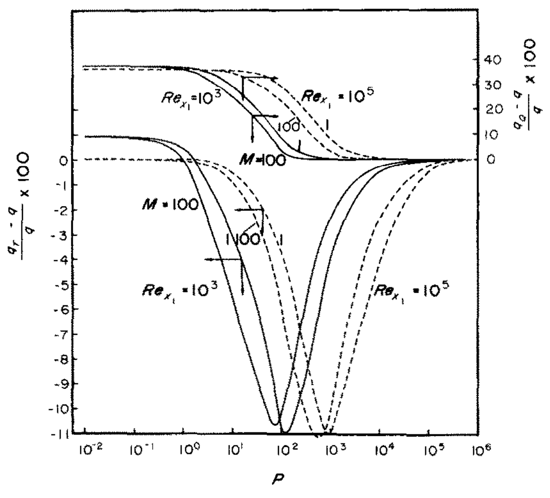

FIG. 6. Comparison of local heat transfer for laminar, cocurrent flow of constant property fluids.

are the same as in the present analysis. At the other extreme, P approaching zero, heat flux predicted on the basis of equation (37) yields a better approximation.

The predictions of this work for the limiting case where one of the sides of the plate is maintained at a constant wall temperature were in excellent agreement with the similarity solution reported by Kuznetsov [5]. The present analysis and results are as valid as the conventional boundary layer theory from which the heat transfer coefficient $h(x, \xi)$, equation (18), was derived. The results are, strictly speaking, correct only when transverse heat conduction in the plate is much greater than the axial heat conduction. This condition is expected to be true when the wall temperature varies only moderately along the plate.

The useful range of equations such as (37) and (38) in predicting the heat exchange between two streams separated by a plane wall can readily be determined for any combination of cocurrent, countercurrent, laminar or turbulent flow by an identical analysis. In a way of generalization, it should be remarked that the present analysis would also be applicable to mass transfer between two fluid streams when there is an interface resistance for diffusion between the phases. Diffusion of water through an evaporation-inhibiting film placed upon a body of water is a specific example of where the present analysis would be applicable.

CONCLUSIONS

The chief value of this analysis is in furthering the understanding of simple cocurrent and countercurrent flow heat (or mass) transfer processes and in enabling the interpretation of data acquired in simple laboratory heat transfer devices. Industrial heat exchange equipment is much more complex than the simple flow models considered here.

The main conclusion of the paper is that the heat transfer coefficients in the individual streams are interdependent, and hence design predictions which neglect this interdependence could differ considerably from reality. The usefulness and the range of validity of local heat-transfer coefficients based on the uniform wall temperature and the uniform heat flux boundary conditions can easily be determined.

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INTERACTION THERMIQUE DE DEUX ÉCOULEMENTS AVEC COUCHE LIMITE SÉPARÉS PAR UNE PLAQUE

Résumé—On considère le problème de l'échange thermique entre deux courants fluides avec couche limite séparés par une plaque plane. On présente une analyse générale applicable aux écoulements cocourants ou contre-courants, laminaires ou turbulents. Une solution exacte pour la distribution de température et le transfert de chaleur le long de la plaque est obtenue dans le cas spécial d'un écoulement à propriétés constantes incompressible et cocourant. Dans le cas moins restrictif d'un écoulement laminaire ou turbulent à propriétés constantes la température pariétale et le flux de chaleur sont déterminés en utilisant la méthode de superposition pour laquelle il est possible d'obtenir un degré de précision voulu. Dans le cas plus général de propriétés physiques variables on indique la solution aux différences finies des équations de quantité de mouvement et d'énergie sous la forme de Von Mises. Finalement, on donne des résultats pour quelques exemples d'écoulements cocourants à propriétés constantes et laminaires. Il est montré que les analyses qui négligent l'interaction thermique entre les courants fluides peuvent conduire á des erreurs importantes.

THERMISCHE WECHSELWIRKUNG ZWEIER STRÖME IN GRENZSCHICHTSTRÖMUNG AN EINER TRENNENDEN PLATTE

Zusammenfassung—Es wird das Problem des Wärmeaustausches zwischen zwei Flüssigkeitsströmen behandelt mit Grenzschichtströmung an einer ebenen Platte, welche die beiden Ströme trennt. Es wird eine allgemeine Betrachtungsweise dargelegt, die anwendbar ist auf Gleich- und Gegenströmung, auf den laminaren und turbulenten Fall. Für den speziellen Fall konstanter Stoffeigenschaften bei zähigkeitsfreier Gleichströmung wurde eine exakte Lösung für die Temperaturverteilung und den Wärmeübergang längs der Platte ermittelt. In dem weniger eingeschränkten Fall laminarer oder turbulenter Strömung mit konstanten Stoffeigenschaften wurde die Wandtemperatur und der Wärmestrom bestimmt mit Hilfe der Superpositionsmethode, womit Ergebnisse ausreichender Genauigkeit zu erzielen sind. Für den allgemeinsten Fall variabler Stoffwerte ist eine Lösung der Impuls- und Energiegleichungen mit einer Differenzenmethode in der von-Mises-Form angegeben. Zum Schluss ist über einige aufschlussreiche Ergebnisse für laminaren Gleichstrom mit konstanten Stoffwerten berichtet. Es wird gezeigt, dass Betrachtungen für den Wärmeaustausch, die die thermischen Wechselwirkungen zwischen den Flüssigkeitsströmen vernachlässigen, ziemlich falsch sein können.

ТЕПЛОВОЕ ВЗАИМОДЕЙСТВИЕ ДВУХ ПОТОКОВ ЖИДКОСТИ, РАЗДЕЛЁННЫХ ПЛАСТИНОЙ

Аннотация—Рассматривается задача теплообмена между двумя потоками жидкости, разделёнными плоской пластиной. Проводится общий анализ, применяемый к спутным или встречным, ламинарным или турбулентным течениям. Получено точное решение для распределения температуры и теплообмена вдоль пластины для специального случая встречного невязкого течения с постоянными свойствами. В более общем случае ламинарного или турбулентного течения с постоянными свойствами температура стенки и тепловой поток рассчитываются по методу наложения, который мохет дать результаты с требуемой точностью. Для более общего случая переменых физических свойств рекомендуется конечно-разностное решение уравнений количества движения и энергии в коэффициентах фон Мизеса. И, наконец, приводятся наглядные данные для спутного ламинарного потока с постоянными физическими свойствами. Показано, что анализ теплообмена в пренебрежении тепловым взаимодействием между потоками жидкости может дать серьёзную ошибку.